Finite Elements in Analysis and Design 168 (2020) 103340

Contents lists available at ScienceDirect

Finite Elements in Analysis and Design

journal homepage: www.elsevier.com/locate/finel

Discussion on "A linear complete extended finite element method for dynamic fracture simulation with non-nodal enrichments" [Finite Elem. Anal. Des. 152 (2018)] by I. Asareh, T.-Y. Kim, and J.-H. Song

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ARTICLE INFO

Keywords:
DE-FEM
NXFEM
XFEM
IGFEM
Weak discontinuities
Strong discontinuities

ABSTRACT

The subject paper purportedly proposes a novel enriched finite element method for modeling problems with strong discontinuities such as those encountered in fracture mechanics. The purpose of this document is to demonstrate that the method in the subject paper (Non-nodal eXtended Finite Element Method, NXFEM) is conceptually identical to the Discontinuity-Enriched Finite Element Method (DE-FEM) [Int. J. Numer. Meth. Eng. 2017; 112:1589–1613] proposed by Aragón and Simone.

1. Introduction

The **Discontinuity-Enriched Finite Element Method (DE-FEM)** [1] is a finite element method pioneered by Aragón and Simone in 2017 that enables the placement of strong and weak enriched nodes along discontinuity lines (*i.e.*, material interfaces and cracks). Asareh, Kim, and Song, in the subject paper [2] published in 2018, propose the **Non-nodal eXtended Finite Element Method (NXFEM)** in which, quoting from the abstract, *"strong and weak discontinuities are assigned to a set of non-nodal points on the interface"*. In this discussion we show that DE-FEM and NXFEM are conceptually identical enriched finite element methods. In this context, enriched FEM is understood as a technique similar to the Generalized Finite Element Method (GFEM) or eXtended Finite Element Method (XFEM) whereby the standard FEM space is augmented by an enriched space that incorporates, through enrichment functions, some *a priori* knowledge about the solution to the problem at hand.

A paradigm alternative to GFEM/XFEM for modeling weak discontinuities using an enriched finite element formulation, whereby enriched degrees of freedom (DOFs) were assigned to nodes created along discontinuities, can be traced back to the work of Soghrati et al. on the Interface-enriched Generalized Finite Element Method (IGFEM) [3]. Since its introduction, IGFEM has been developed in many directions to handle three-dimensional problems [4], multiple discontinuities interacting within a single finite element through the Hierarchical Interface-enriched FEM (HIFEM) [5,6], curved interfaces [7], NURBS discontinuities [8–11], immerse boundary/fictitious domain problems [12], and shape optimization [13–16]. In addition, IGFEM/HIFEM has been successfully applied to numerous engineering problems [17–30].

By building on previous work on IGFEM/HIFEM [3–30], our article [1] introduces a finite element method that we named DE-FEM. This novel method makes use of enrichment functions that reproduce both weak and strong discontinuities with the same formulation. For the first time, DE-FEM collocates the enriched nodes associated with strong and weak enrichment functions along discontinuities (at their intersection with edges of finite elements), rather than at existing discretization nodes of elements cut by discontinuities as it is done in GFEM/XFEM. DE-FEM therefore provides a generalization of IGFEM/HIFEM for treating problems with both weak and strong discontinuities; in the absence of the latter, the formulation recovered is that of IGFEM/HIFEM.

As demonstrated in this discussion document, the Moëst salient aspects of the method presented in the subject paper [2], submitted August 12th, 2018, are identical to those presented in our

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https://doi.org/10.1016/j.finel.2019.103340
Received 15 January 2019; Received in revised form 23 September 2019; Accepted 8 October 2019
Available online XXX

paper, entitled "The Discontinuity-Enriched Finite Element Method (DE-FEM)" [1], published online April 13th, 2017, in the *International Journal for Numerical Methods in Engineering* (IJNME).

2. On the similarities between the two methods

At a glance, both methods share the following properties:

i) Enriched DOFs are not associated with the nodes of the original mesh but with nodes collocated at the intersection points between discontinuities and the edges of the background finite element mesh;
ii) At the location of a newly created enriched node, two sets of DOFs are assigned to represent the jumps in the field (strong DOFs) and its gradient (weak DOFs);
iii) The enrichment functions are non-zero only in cut elements (therefore local by construction)—this avoids problems in blending elements (elements that are contiguous to cut elements);
iv) Because of *iii)*, the partition of unity of the background mesh is not required to localize enrichment functions in cut elements as it is usually done in GFEM/XFEM;
v) Because of *iv)*, formulation and corresponding computer implementation are simpler than those of GFEM/XFEM;
vi) Enriched DOFs associated to the strong enrichment function represent the actual displacement jump at the enriched node location and have therefore a direct physical interpretation;
vii) Enrichment functions are identically zero at the nodes of the original mesh by construction. This has two important consequences: *a)* Prescribing essential (Dirichlet) boundary conditions on original mesh nodes is as straightforward as in standard FEM; and *b)* DOFs associated to mesh nodes retain their physical meaning, *i.e.*, the displacement at the node location in displacement-based formulations;
viii) Enriched DOFs—both weak and strong—can also be prescribed directly as in standard FEM; and finally
ix) Both methods adopt a Bubnov-Galerkin approach for the discretization of the weak (variational) form, whereby the trial solution and the weight function are composed by a three-term approximation (more on this crucial point in § 4).

3. On the similarities with respect to IGFEM

It is worth noting that points *i)*, *iii)*, *iv)*, *v)*, *vii)*, and *viii)* in § 2 are also similarities shared with IGFEM [3] if one makes reference to the weak DOFs alone. In our paper on DE-FEM, we discussed previous work on IGFEM and stated that we develop on it. The name Discontinuity-Enriched Finite Element Method was given because, from the technological point of view, DE-FEM could accomplish what IGFEM could not: the correct reproduction of the kinematics that results from having a strong discontinuity, for which the formulation required an additional enriched term. In other words, just changing the IGFEM enrichment function from weak to strong, as presented in Ref. [31] by the authors of the subject paper, yields a formulation with an insufficient number of DOFs to reproduce two independent kinematic fields on either side of a discontinuity, as discussed next.

Consider a one-dimensional bi-material bar discretized with a single finite element. The corresponding IGFEM approximation is given by

$$u^h(x) = \underbrace{\sum_{j=1}^{2} N_j(x)\, U_j}_{\text{std. FEM}} + \underbrace{\psi(x)\,\alpha}_{\text{enrichment}}, \tag{1}$$

where ψ is a C^0 enrichment function that captures the jump in the gradient at the bi-material interface. From all possible C^0–continuous functions, IGFEM uses a piecewise linear function that is constructed with the aid of Lagrange shape functions in integration elements (subdomains that are created for the purpose of numerical quadrature of the element's stiffness matrix and force vector). Thus, in the spirit of GFEM, IGFEM uses *a priori* information about the problem at hand to construct the enrichment functions. This point is further discussed in § 4. If instead of a bi-material interface we have a crack, a first approach would be to replace ψ with another enrichment function that exhibits a jump in the field (a C^{-1}–continuous function). Such formulation is however flawed, since there are not enough DOFs to represent the kinematics properly. For our one-dimensional example we would need four DOFs instead of three as given by (1). In two dimensions this issue is revealed by performing a simple discontinuous patch test.

The approximation in (1) with a C^{-1}–continuous enrichment function was in fact our first attempt (submitted May 31st, 2016 to IJNME) to resolve strong discontinuities by collocating enriched nodes along discontinuities. Aragón presented this formulation for the first time in Seoul during the 12th World Congress on Computational Mechanics, July 24–29, 2016. The same formulation appeared in Asareh et alii's first publication on the subject [31] (submitted March 24th, 2018).

4. On discrete finite element spaces

The crux of the method lies in how the discrete finite element spaces are built.

4.1. One-dimensional case

In [2, Eq. (4)], for a cut one-dimensional element, the approximation takes the form

$$u(X,t) = \underbrace{\sum_{I=1}^{2} N_I(X) u_I(t)}_{\text{std.FEM}} + \underbrace{\underbrace{\Psi^u(X)[\![u(t)]\!]}_{\text{strong}} + \underbrace{\Psi^{\nabla u}[\![L^e \nabla u(t)]\!]}_{\text{weak}}}_{\text{enrichment}}, \tag{2}$$

where the first term represents the standard FEM component, and the other terms enrich the approximation to deal with strong (second term) and weak (third term) discontinuities.

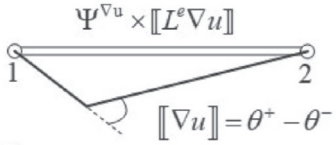
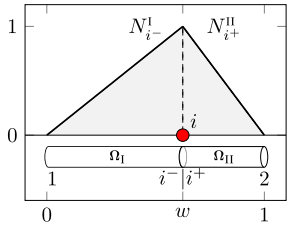
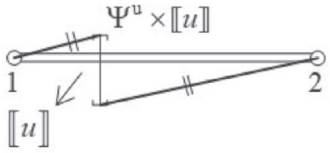
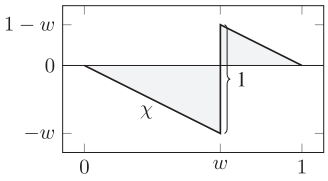
In DE-FEM [1], the approximation for a one-dimensional element takes the form

$$u^h(x) = \underbrace{\sum_{j=1}^{2} N_j(x)\, U_j}_{\text{std. FEM}} + \underbrace{\underbrace{\psi(x)\,\alpha}_{\text{weak}} + \underbrace{\chi(x)\,\beta}_{\text{strong}}}_{\text{enrichment}}, \tag{3}$$

which is identical to (2) since the standard part of the approximation (the first term) is enriched by two additional terms that deal with the weak and strong discontinuities—except that the enriched terms are given in different order and a different notation is used. While the notation used for the first term is the same in both equations, DOFs α and β are equivalent to $[\![L^e \nabla u(t)]\!]$ and $[\![u(t)]\!]$, respectively.

Not only the equations for the approximation are identical, their corresponding enriched finite element spaces are linear and therefore also identical, yielding identical solutions. Table 1 presents an overview of the enrichment functions used, where it can be seen that the negative of our weak enrichment scaled is proposed in Ref. [2]. The authors have used scaling factors s^+ and s^- so that the weak enrichment function vanishes as the discontinuity approaches the standard nodes, just as it is done in IGFEM/HIFEM works [3–7, 10, 17, 19, 20]. Scaling the weak enrichment functions has the effect of improving the condition number of the stiffness matrix, an issue that is well known in enriched FEM. In our DE-FEM paper [1] the weak enrichment function was not scaled because the issue of the condition number had been discussed in previous works [3, 7]. However, in § 2.1 we state that *"… the weak enrichment attains a maximum value of one at* x_Γ*, and it ramps linearly to zero at the ends of the bar. Note that the IGFEM weak enrichment uses a scale factor to improve the conditioning as the interface is moved closer to a node; to ease the presentation, such scaling is omitted here."*

Table 1
Comparison between the enrichment functions for a one-dimensional cut element.

NXFEM [2]	DE-FEM [1]
Weak enrichment $\Psi^{\nabla u}$ (Eq. (7) and Fig. 2(d) in [2])	Weak enrichment ψ (Eq. (3) and Fig. 2(b) in [1]):
$\Psi^{\nabla u} = \begin{cases} -s^+ N_2 & f < 0, \\ -s^- N_1 & f > 0, \end{cases}$ where $s^- = (X^c - X_1)/L^e$ and $s^+ = (X_2 - X^c)/L^e$.	$\psi = \begin{cases} \frac{x}{x_\Gamma} & \text{for } x < x_\Gamma \\ \frac{L-x}{L-x_\Gamma} & \text{for } x > x_\Gamma \end{cases}$.
	
Strong enrichment function Ψ^u (Eq. (5) and Fig. 2(c) in [2]):	Strong enrichment function χ (Eq. (7) and Fig. 2(c) in [1]), which can be expressed in terms of the partition of unity shape functions after replacing $w = x_\Gamma/L$, *i.e.*, the location of the discontinuity relative to the length:
$\Psi^u = \begin{cases} -N_2 & f < 0, \\ N_1 & f > 0, \end{cases}$ where $f > 0$ represents the positive side of the discontinuity.	$\chi = \begin{cases} -w\frac{x}{x_\Gamma} = -N_2 & \text{for } x < x_\Gamma \\ (1-w)\frac{L-x}{L-x_\Gamma} = N_1 & \text{for } x > x_\Gamma \end{cases}$.
	

The strong enrichment proposed in the subject paper is the negative of ours, as it can be seen by simply comparing the figures. Yet, in the one-dimensional case the resulting space is the same and so are the solutions obtained.

4.2. Two-dimensional case

In two dimensions, the equation that defines the method remains the same: in [2, Eq. (8)]

$$u(\mathbf{X}) = \underbrace{\sum_{I=1}^{3} N_I(\mathbf{X}) u_I}_{\text{std.FEM}} + \underbrace{\underbrace{\sum_{J\in\delta^u} \Psi^u_J(\mathbf{X}) [\![u]\!]_J}_{\text{strong}} + \underbrace{\sum_{K\in\delta^{\nabla u}} \Psi^{\nabla u}_K [\![L^c_K \nabla u \cdot \mathbf{e}^c_K]\!]_K}_{\text{weak}}}_{\text{enrichment}}, \tag{4}$$

while in our work

$$\boldsymbol{u}^h(\boldsymbol{x}) = \underbrace{\sum_{i\in\iota_h} N_i(\boldsymbol{x}) \boldsymbol{U}_i}_{\text{std.FEM}} + \underbrace{\underbrace{\sum_{i\in\iota_w} \psi_i(\boldsymbol{x}) \boldsymbol{\alpha}_i}_{\text{weak}} + \underbrace{\sum_{i\in\iota_s} \chi_i(\boldsymbol{x}) \boldsymbol{\beta}_i}_{\text{strong}}}_{\text{enrichment}}, \tag{5}$$

where, as in the one-dimensional case, the three-term equations are identical.

The enrichment functions for a cut triangular element are however different on one side of the discontinuity. In the subject paper, the strong enrichment functions [2, Eq. (9)]

$$\Psi^u_1 = \begin{cases} -\frac{N_1 \times N_3}{N_2 + N_3} & f < 0, \\ N_3 & f > 0, \end{cases} \quad \Psi^u_2 = \begin{cases} -\frac{N_1 \times N_2}{N_2 + N_3} & f < 0, \\ N_2 & f > 0, \end{cases} \tag{6}$$

and the weak enrichment functions [2, Eq. (11)]

$$\Psi^{\nabla u}_1 = \begin{cases} -s^+_1 \frac{N_1 \times N_3}{N_2 + N_3} & f < 0, \\ -s^-_1 N_3 & f > 0, \end{cases} \quad \Psi^{\nabla u}_2 = \begin{cases} -s^+_2 \frac{N_1 \times N_2}{N_2 + N_3} & f < 0, \\ -s^-_2 N_2 & f > 0 \end{cases} \tag{7}$$

are used. Strong and weak enrichment functions (6) and (7) are shown, respectively, in Figs. 1 and 2. Although the functions are linear in the positive side of the cut ($f > 0$, region above the discontinuity line in the figure), they are nonlinear in the negative side and, as shown in the figures, the nonlinearity dramatically increases as crack intersection points move further away from their respective edge centers—the nonlinearity is also confirmed in Ref. [2, § 3.2]. The weak enrichment functions are discontinuous for a general crack position as also noted in Ref. [2, § 2.2].

In [1] we propose a set of rules for building the enrichment functions not only for triangular (T3) but also for quadrangular (Q4) cut elements. For the latter, the enrichment functions are straightforward as long as Q4s are cut into quadrangles (with enriched nodes located at opposite edges). The case of a Q4 cut through contiguous edges requires more attention because the enrichment functions should be able to recover the bilinear behavior of the parent element, *i.e.*, the xy term. A simple solution was proposed in [1] between Eqs. (27) and (28), and the issue is further discussed in a recently published paper [32].

Because the subject paper only deals with T3s, we now focus entirely on these elements. The enrichment functions proposed in the subject paper are essentially identical on one side of the crack (the triangular part) and different on the other side. In DE-FEM, the enrichment functions for T3s are linear not only on the positive side of the discontinuity (as those of the subject paper), but remain linear on the negative side as well. We show [1, § 3.1] that our formulation, with the use of linear enrichments, passes the discontinuous patch test and thus is able to correctly describe two independent kinematic fields with constant stress at each side of a crack.

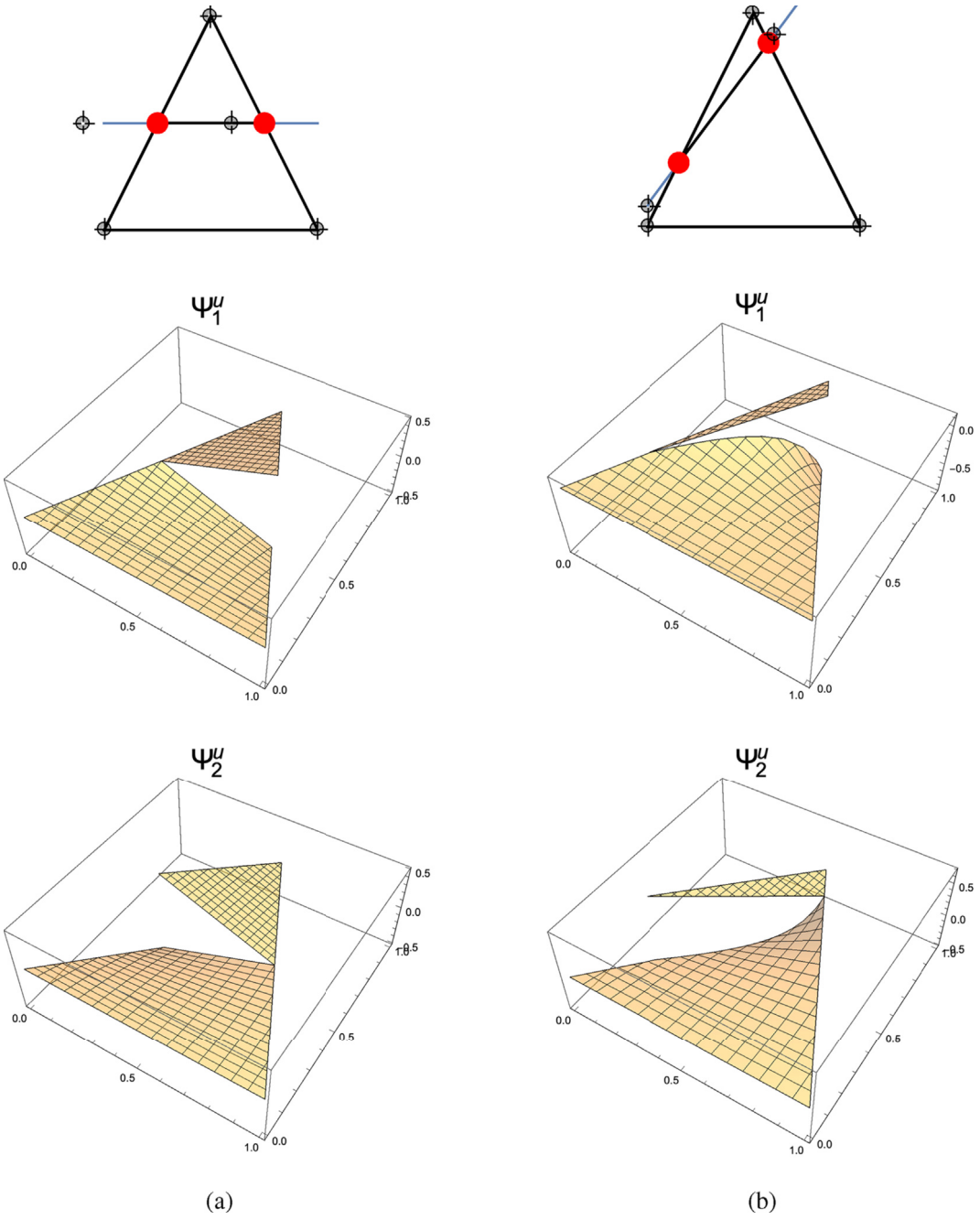


Fig. 1. Strong enrichment functions (6) for a discontinuity intersecting element edges (a) at the center and (b) offset from the center. The left-hand side column images are directly comparable to the top row images of Fig. 7 in Ref. [2].

Comparing the two methods for the two-dimensional case, it can be concluded that:

- Although the subject paper provides proof that NXFEM enrichment functions can recover two linear independent fields at each side of the crack, the nonlinear rational functions proposed are unnecessarily complex, more expensive, and not exactly integrable by means of standard Gauss quadrature.[1] Proving linear completeness in DE-FEM is irrelevant because our enrichment functions are constructed from Lagrange shape functions in integration elements, which were devised to be linearly complete in the first place. As a result, all enrichment functions in DE-FEM can be integrated exactly.
- The NXFEM weak enrichment function is discontinuous within the element, which not only has implications in the computation of the displacement jump within the element—as it cannot be inferred by only considering strong DOFs alone as in DE-FEM—but also undermines the physical meaning attributed to weak DOFs. Furthermore, because of its discontinuous nature, the weak enrichment function cannot be used as is to model problems that are globally C^0–continuous (*e.g.*, material interfaces). In DE-FEM, dropping the strong enrichment term yields the IGFEM/HIFEM formulation

[1] Note that the proof on the reproduction of independent linear fields in [2, Appendix B] contains a mistake in describing the weak jump conditions in Eq. (B.1) that leads to a different coefficient matrix in Eq. (B.2).

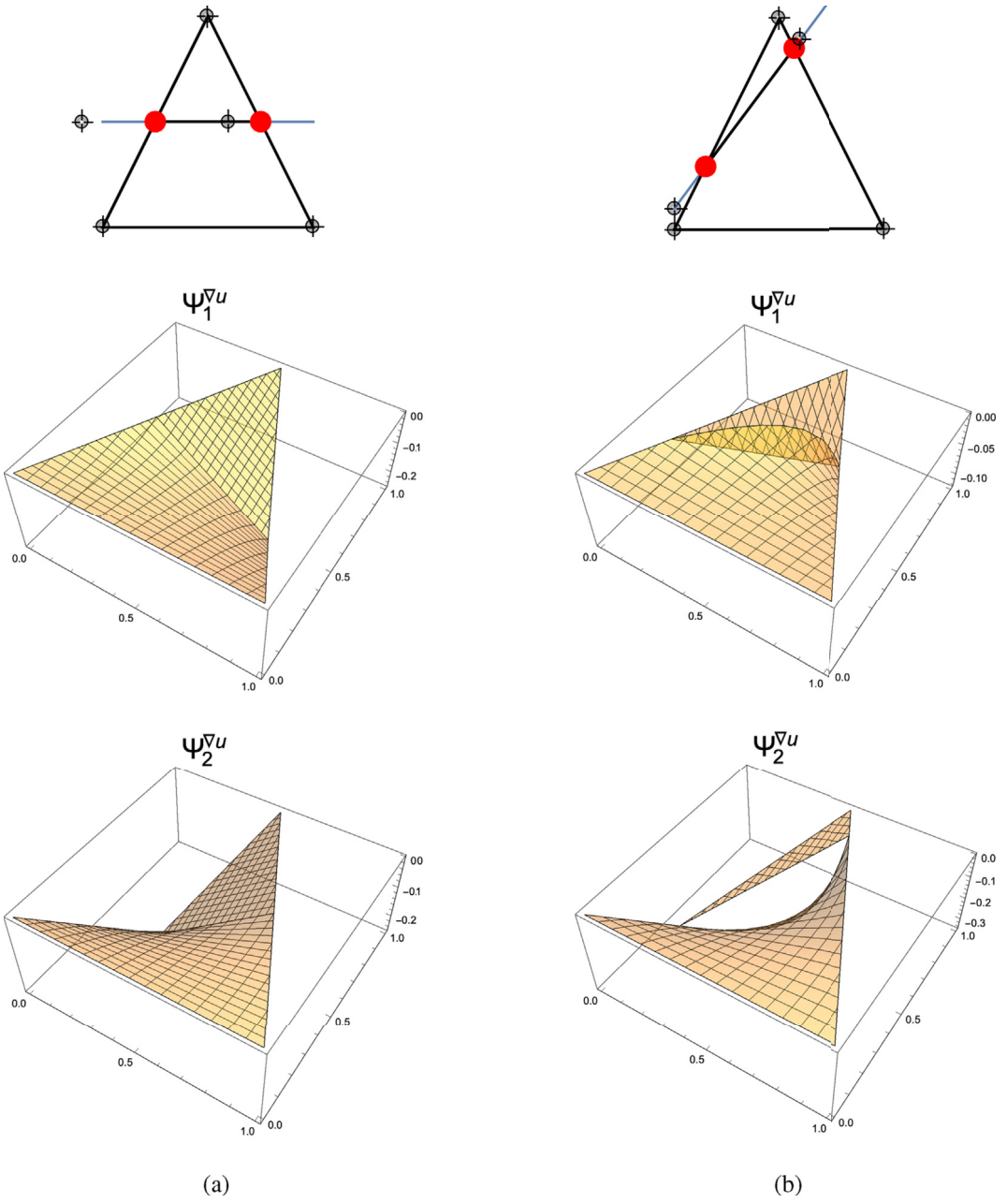


Fig. 2. Weak enrichment functions (7) for a discontinuity intersecting element edges (a) at the center and (b) offset from the center. The left-hand side column images are directly comparable to the center row images of Fig. 7 in Ref. [2].

that was devised exclusively for modeling problems containing only weak discontinuities;

- Finally, since NXFEM enrichment functions (and integration scheme) rely on the entire splitting of an element, NXFEM cannot describe a crack tip within an element while DE-FEM can [1, Fig. 7].

The enrichment functions proposed in the DE-FEM paper are clearly superior under all aspects.

The NXFEM enrichment functions proposed in the subject paper have been derived from a procedure [31, § 2.2] whose main ingredients—satisfying function jump conditions and C^0–continuity between enriched and contiguous elements—had already been used for creating enrichments in IGFEM and DE-FEM works [1,3–30]. The NXFEM procedure, however, adds one (unnecessary) extra requirement: All enriched DOFs have to directly represent physical quantities of the problem. As a side note, a procedure to obtain enrichment functions does not constitute a new method since enrichments are a mere accessory to the underlying finite element technology, which was in this case pioneered in IGFEM and extended in DE-FEM.

4.3. On the construction of DE-FEM enrichment functions

There are no strict requirements on how to build the enriched part of the approximation as long as enrichment functions improve on the solution (note the validity of Céa's approximation theorem according to which the finite element solution minimizes the error in the FEM discrete space). Weak and strong enrichment functions in DE-FEM were designed to reproduce the correct kinematics of weak and strong discontinuities, respectively. Linear Lagrange shape functions are not only

perfectly valid candidates to do so but also the simplest and Moëst rational choice.

The enrichment component in DE-FEM is built by using Lagrange shape functions of integration subdomains and this proceduce should not be confused with (hidden) remeshing. Remeshing is the set of procedures that modify the original background finite element discretization in any way, *e.g.*, by changing the nodal coordinate and/or element connectivity arrays. Referring back to (5), remeshing would imply that the standard FEM part of the approximation changes. Since this component is kept intact in DE-FEM, there is no hidden remeshing.

5. On the derivation of DE-FEM from GFEM/XFEM

The three-term DE-FEM approximations (3) and (5), or their equivalent expressions (2) and (4), respectively, have been given as the starting point for the derivation of the discrete set of finite element equations. The main differences between DE-FEM and GFEM/XFEM are the use of the mesh partition of unity to localize enrichment functions and the total number of DOFs, which is higher in GFEM/XFEM because enrichments are assigned to existing mesh nodes. In this discussion document, for the first time, we show that it is possible to derive the three-term approximation from GFEM/XFEM by a proper choice of enrichment functions and subsequent clustering in order to reduce the number of DOFs.

Consider a one-dimensional linear finite element with nodes x_1 and x_2 that contains a discontinuity at x_Γ. For this element, the GFEM/XFEM approximation can be written as

$$u^h(x) = \underbrace{\sum_{i=1}^{2} N_i(x)\, U_i}_{\text{std. FEM}} + \underbrace{\sum_{i=1}^{2} N_i(x) \sum_{j=1}^{2} E_{ij}(x)\, \hat{U}_{ij}}_{\text{enrichment}}, \tag{8}$$

where we have chosen to use two enrichment functions E_{ij} to represent both weak and strong discontinuities. The functions in the enrichment term in (8) can be clustered following the work of Duarte et al. [33] to obtain the enrichment in the DE-FEM approximation. To wit,

$$\sum_{i=1}^{2} N_i \sum_{j=1}^{2} E_{ij}\hat{U}_{ij} = \underbrace{(N_1E_{11} + N_2E_{21})}_{\psi}\alpha + \underbrace{(N_1E_{12} + N_2E_{22})}_{\chi}\beta, \tag{9}$$

where $\hat{U}_{11} = \hat{U}_{21} = \alpha$ and $\hat{U}_{12} = \hat{U}_{22} = \beta$ denote the enriched DOFs associated to, respectively, the clustered enrichment functions (*cf.* Fig. 3)

$$\psi = N_1E_{11} + N_2E_{21} = N_1c_1\mathcal{H}(x - x_\Gamma) + N_2c_2\mathcal{H}(x_\Gamma - x), \tag{10}$$

and

$$\chi = N_1E_{12} + N_2E_{22} = N_1c_3\mathcal{H}(x - x_\Gamma) + c_4N_2\mathcal{H}(x_\Gamma - x). \tag{11}$$

For the weak enrichment function (10), the constants $c_1 = 1/(1 - w)$ and $c_2 = 1/w$, with $w = x_\Gamma/L$, yield a C^0–continuous function that attains a maximum value of one regardless of the discontinuity location within the element, as shown in Table 1 (top-right). Similarly, choosing $c_3 = 1$ and $c_4 = -1$ in the strong enrichment function (11) results in the enrichment shown in Table 1 (bottom-right). Equations (10) and (11), however, represent a family of enrichment functions that depends on different coefficients c_i (which are at Moëst function of discontinuity location). This procedure is analogous in two dimensions.

For this one-dimensional example, it is only necessary to change the coefficients c_1 through c_4 to recover the enrichment functions used in the subject paper: While the strong enrichment can be

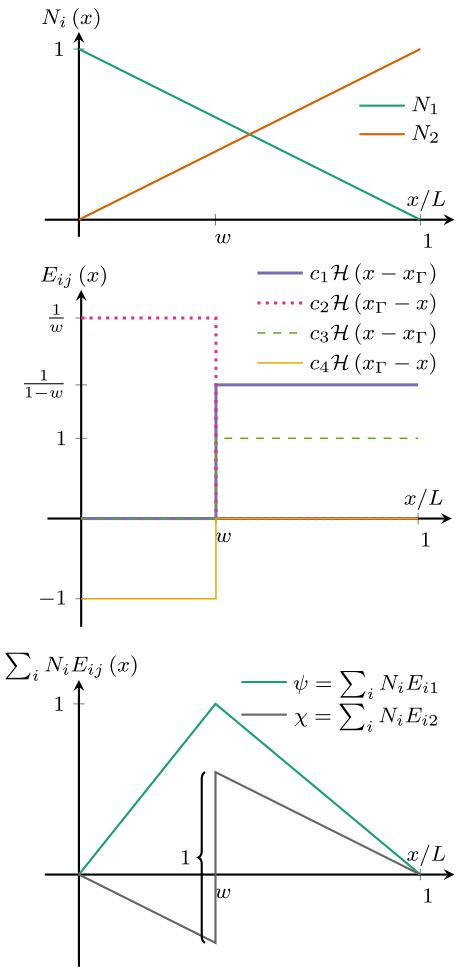


Fig. 3. Construction of the weak and strong DE-FEM enrichment functions (bottom) from standard FEM shape functions (top) and GFEM/XFEM enrichment functions (center). In the figure, $w = x_\Gamma/L$.

obtained by choosing $c_3 = -1$ and $c_4 = 1$ (the negative of our constants), the weak enrichment is recovered by choosing $c_1 = -w$ and $c_2 = w - 1$. *Mutatis mutandis*, analogous considerations can be made for the IGFEM formulations and the method presented in Ref. [31].

6. On the differences

The dissimilarities between NXFEM and DE-FEM are not essential to the underlying functioning principles of the methods. In the subject paper [2], NXFEM is applied in a dynamic setting, whereas the DE-FEM paper [1] only deals with elastostatics. Nevertheless, as explained above, the finite dimensional approximation is identical in both papers if the same enrichments are adopted, and the choice of the dynamic setting could be considered at the same level of an application of the method for the problem at hand since it does not imply any conceptual complication. In both works the positive and negative regions in the vicinity of a discontinuity are endowed with a normal vector field, but NXFEM uses a level-set function to define them. Also, a crack tip in NXFEM is defined by constraining both DOFs to zero (see Eq. (15) in Ref. [2]), contrary to DE-FEM where only the strong DOF is set to zero and the Galerkin projection determines the value of weak DOFs.

Other technicalities that are not essential to the underlying finite element technology are discussed next.

6.1. On the physical interpretation of DOFs

Weak and strong enriched DOFs in NXFEM are explicitly assigned a physical interpretation, and the standard notation for symbols commonly used to denote enriched (or generalized) DOFs is replaced by symbols that denote *"jumps in the displacement and strain fields"* at the location of enriched nodes: $[\![u]\!]_J$ and $[\![L_K^c \nabla u \cdot \mathbf{e}_K^c]\!]_K$ for strong and weak DOFs, respectively.

It is worth noting that the physical meaning attributed to the weak DOFs makes sense as long as the weak enrichment function is C^0–continuous, which is true only at the location of enriched nodes but not within the element as shown in Fig. 2. Note in the figure that the discontinuous nature of the functions increases as crack intersection points are located further away from the edge center. In Ref. [2] it is stated that *"for a general crack position, the weak enrichment functions $\Psi^{\nabla u}$ may introduce discontinuities across the interface."* and later it is stressed that *"the derivative enrichment functions $\Psi^{\nabla u}$ are not in general continuous across the interface. So, they contribute to cohesive force computations."* Therefore, because weak enrichment functions are in general discontinuous, the diplacement jump along the discontinuity within an element is obtained by a combination of weak and strong DOFs, undermining the bestowed physical interpretation. This is indeed confirmed in the subject paper since the displacement jump is obtained after evaluating the enrichment part of the interpolation (both weak and strong terms, as per Eq. (22) in [2]):

$$\delta_N = \mathbf{n} \cdot [\![\mathbf{u}]\!] = \mathbf{n} \cdot \left(\sum_{J \in \delta^u} [\![\Psi_J^u]\!][\![\mathbf{u}]\!]_J + \sum_{K \in \delta^{\nabla u}} [\![\Psi_K^{\nabla u}]\!][\![L_K^c \nabla \mathbf{u} \cdot \mathbf{e}_K^c]\!]_K \right).$$

In DE-FEM, on the other hand, strong DOFs at enriched node locations can fully describe the displacement jump not only at those points but also along the entire discontinuity segment within the cut element. More specifically, the strong enrichment function was designed with the aid of the partition of unity shape functions so that the jump at the location of enriched node $\boldsymbol{x}_j$ is equal to one (in fact, $[\![\chi_i(\boldsymbol{x}_j)]\!] = \delta_{ij}$ with δ_{ij} denoting the Kronecker delta, and thus $\boldsymbol{\beta}_j$ in (5) physically represents the jump in the displacement field at that location). And because the weak enrichment function is C^0–continuous within the element and varies linearly between enriched nodes $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$, the displacement jump along the corresponding discontinuity segment can be readily obtained by linear interpolation of the corresponding strong DOFs $\boldsymbol{\beta}_i$ and $\boldsymbol{\beta}_j$ (per Eq. (31) in [1]). Therefore, evaluating the jump along a discontinuity is trivial in DE-FEM and makes the evaluation of cohesive forces straightforward. Regarding weak DOFs, we do not explicitly endow them with a physical meaning because we scale their corresponding weak enrichment functions in order to preserve stability, *i.e.*, to prevent that the condition number of the stiffness matrix grows unbounded. Noteworthy, applying a Jacobi-like preconditioner can effectively deal with ill-conditioning regardless of whether enrichment functions are scaled or not [32].

6.2. On numerical quadrature

One difference, which again is not fundamental for the underlying principles of the method, lies in the way numerical integration of cut elements' local arrays is performed. One could use adaptive quadrature, whereby integration is conducted on adaptively refined subdomains of the original uncut mesh element until accuracy is reached within tolerance. However, because of the enrichment functions' non-smooth discontinuous nature, creating integration elements performs better [34]. Indeed, because sides of integration elements match to discontinuities, enrichment functions within each integration subdomain are smooth and thus can be integrated exactly using the least number of quadrature points.

The use of integration elements for the quadrature of discontinuous functions was one of the ideas put forward in the first XFEM paper [35, § 3.3]. This practice has been ubiquitous in the field ever since (see for instance [34,36,37]) and it is what we do in DE-FEM. In addition, integration elements in DE-FEM serve other two purposes: They aid in constructing the enrichment functions for the approximation as discussed in § 4.3 and also aid in post-processing so that the displacement field (*e.g.*, the crack opening) can be visualized correctly.

In [2], integration subdomains are created as well (see § 3.2 and Fig. 8). However, two integration schemes based on the parent domain are used. For a quadrangular parent domain (used for enrichment functions [2, Eqs. (10) and (12)]), straight interfaces in physical space become curves in parent domain because of the Duffy mapping, and *"to maintain the accuracy of the integration, the interface is discretized, which leads to increased number of subdomains"*. For the triangular parent domain (used for enrichment functions [2, Eqs. (9) and (11)]), straight interfaces remain straight, but because *"enrichment functions are non-polynomial functions, higher number of Gauss points are required to maintain the integration accuracy."* As a result, although subdomains are created for integration purposes, the procedure proposed in the subject paper necessitates many more integration points than those required in DE-FEM or XFEM either because of the non-polynomial nature of the enrichment functions or because of the Duffy mapping.

7. Concluding remarks

Are DE-FEM and NXFEM two different methods? A superficial analysis of the technical differences between the two papers could lead to consider DE-FEM and NXFEM as two different methods. Another view on this issue can be found in the corrigendum [38] to the subject paper [2] where it is stated that our DE-FEM paper [1] is a *"previously published work that exhibits similar concepts"*. We disagree with both views.

Table 2
Publication timeline and enrichment type used.

ARTICLE OR ABSTRACT	NAME OR ACRONYM	ENRICHMENT TYPE	(SUBMITTED) PUBLISHED ONLINE	CONFERENCE DATES
Soghrati, Aragón, Duarte, and Geubelle [3]	IGFEM	weak	(Mar. 8th, 2011) Aug. 25th, 2011	
Asareh [41]	(not known)	strong (assumed)		May 22–25, 2016
Aragón and Simone [42]	d_e-GFEM[a]	strong		Jul. 24–29, 2016
Aragón and Simone [43]	d_e-GFEM[a]	strong		Oct. 10–12, 2016
Aragón and Simone [1]	DE-FEM	strong and weak	(May 31st, 2016) Apr. 13th, 2017	
Asareh and Yoon [44]	(not known)	strong (assumed)		Jun. 4–7, 2017
Aragón and Simone [45]	DE-FEM	strong and weak		Jun. 19–21, 2017
Aragón and Simone [46]	DE-FEM	strong and weak		Jul. 17–20, 2017
Asareh, Yoon, and Song [31]	Non-nodal XFEM	strong	(Mar. 24th, 2018) Jul. 5th, 2018	
Asareh, Kim, and Song [2]	NXFEM	strong and weak	(Aug. 12th, 2018) Oct. 5th, 2018	

[a] Stands for "discontinuity-enriched Generalized Finite Element Method".

As demonstrated in this discussion, DE-FEM and NXFEM are two identical FEM methods that enable the placement of strong and weak DOFs along a discontinuity line. The only essential component of these two methods is the use of a three-term approximation of the form shown in § 4. The only relatively important difference between the methods lies in the choice of the two-dimensional weak and strong enrichment functions at one side of the discontinuity (and the NXFEM enrichment functions show no concrete advantage compared to ours, as discussed in § 4.2, and are in fact more complex than necessary). It is also important to stress that this difference does not suffice to make DE-FEM and NXFEM conceptually different methods. By way of example, although XFEM was first introduced for strong discontinuities [35], enrichment functions for modeling weak discontinuities were introduced later and the name of the method remained the same [39,40]. Testing for a different enrichment function by no means justifies changing the name of the method to something else.

Table 2 lists all the NXFEM-related work prior to the online publication of the subject paper, together with key journal publications and our conference abstracts and presentations related to the DE-FEM paper [1]. The timeline and content of the works listed in Table 2 show that the method under discussion was firstly proposed by Aragón and Simone [1] who named it DE-FEM. Subsequent work is unlikely to be classified as independent discovery or co-invention.

Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.finel.2019.103340.

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